Derivation of the Fokker-Planck Equation

Master equation and Kramers-Moyal expansion

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June 12, 2019

Assume we have n space states $x_1, \ldots, x_n$ and at a fixed time t the probability of being in a spate x_i is give by $p_i(t)$. Let the probability transition rate matrix be $\mathbf{T}$ such that $\frac{d\mathbf{p}}{dt} = \mathbf{T}\mathbf{p}(t)$. This means that for each state i, the time rate of increase in probability depends on the contribution from all other states to state i as

$$\frac{dp_i(t)}{dt} = \sum_{j=1}^{n} T_{ij} p_j(t) \tag{1}$$

$$\begin{aligned}\implies \sum_{i=1}^{n} \frac{dp_i(t)}{dt} &= \sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} p_j(t) \\ &= \sum_{j=1}^{n} p_j(t) \sum_{i=1}^{n} T_{ij} \qquad (2)\end{aligned}$$

Since $\sum_{i=1}^{n} p_i(t) = 1$ for all probability vectors, therefore we have $\sum_{i=1}^{n} T_{ij} = 0 \ \forall \ j \in [n]$. Equation (1) can be again written as

$$\begin{aligned}\frac{dp_i(t)}{dt} &= \sum_{j=1}^{n} T_{ij} p_j(t) \\ &= \sum_{j=1, j\neq i}^{n} T_{ij} p_j(t) + T_{ii} p_i(t) \\ &= \sum_{j=1, j\neq i}^{n} T_{ij} p_j(t) - T_{ji} p_i(t) \qquad (3)\end{aligned}$$

Equation (3) is called the master equation as it describes the time evolution of a system that can be modeled as probabilistic combination of states at any given time. This can even be generalized to infinite state space, for example the real line.

We can now generalize the transition rate matrix to a transition kernel T which is a function of 2 states, and $p(\cdot, t)$ shall denote the density function over the continuous state space at time t. Equation (3) will now become

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{\infty} \left[T(x,y)p(y,t) - T(y,x)p(x,t)\right] dy \tag{4}$$

For convenience, let us redefine the transition kernel as a function of the jump $r := x - y$ as $T(y, r) \leftarrow T(x, y)$. Equation (4) can now be written like

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{\infty} T(x-r, r)p(x-r, t)dr - p(x,t) \int_{-\infty}^{\infty} T(x, -r)dr \tag{5}$$

We can now use a Taylor series to write the term in the first integral in Equation (5) as

$$T(x-r,r)p(x-r,t) = T(x,r)p(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} r^n \frac{\partial^n}{\partial x^n} \left[T(x,r)p(x,t)\right] \tag{6}$$

Plugging back Equation (6) in Equation (5) we get

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial t} &= \int\limits_{-\infty}^{\infty} \left[T(x,r)p(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} r^n \frac{\partial^n}{\partial x^n} \left[T(x,r)p(x,t)\right]\right] dr - p(x,t) \int\limits_{-\infty}^{\infty} T(x,-r) dr \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left\{ \left[\int\limits_{-\infty}^{\infty} r^n T(x,r) dr \right] p(x,t) \right\}
\end{aligned} \tag{7}$$

Lets define $a_n(x) := \int_{-\infty}^{\infty} r^n T(x,r) dr$ and one gets the Kramers-Moyal expansion of the master equation

$$\frac{\partial p(x,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[a_n(x) p(x,t)\right] \tag{8}$$

where $a_n(x)$ as per the originally defined transition kernel (a function of 2 states) is therefore

$$a_n(x) := \int_{-\infty}^{\infty} (y-x)^n T(y,x) dy \tag{9}$$

Equation (8) for the special case $n = 2$ is known as the Fokker-Planck equation. The first term is then called the *drift* and the second term is called the *diffusion* term, while $a_1(x)$ and $a_2(x)$ are called diffusion coefficients.

The Puwala theorem states that the Equation (8) either stops at the first term of the second term. If the expansion continues past the second term, it must contain an infinite number of terms in order that the solution to the equation be interpretable as a probability density function.